

# Partially asymmetric exclusion models with quenched disorder

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We consider the one-dimensional partially asymmetric exclusion process with random hopping rates, in which a fraction of particles (or sites) have a preferential jumping direction against the global drift. In this case the accumulated distance traveled by the particles,  $x$ , scales with the time,  $t$ , as  $x \sim t^{1/z}$ , with a dynamical exponent  $z > 0$ . Using extreme value statistics and an asymptotically exact strong disorder renormalization group method we exactly calculate,  $z_{pt}$ , for particlewise (pt) disorder, which is argued to be related as,  $z_{st} = z_{pt}/2$ , for sitewise (st) disorder. In the symmetric case with zero mean drift the particle diffusion is ultra-slow, logarithmic in time.

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Driven lattice gas models are able to describe the generic features of stochastic transport far from equilibrium [1, 2, 3]. This kind of transport is often observed, e.g. in transport systems [4] or (micro-)biological applications [5], where a steady flow is maintained by a steady input of energy. As an important recent application we mention the bi-directional transport on microtubules [6], in which motion of cargos is realized by attaching a few molecular motors of different type. In our approach a cargo-motor complex is modeled as an effective particle with random hopping rates, the value of which depends on the number of attached motors. This kind of molecular machine allows for an efficient navigation in the cell, because the velocity can be controlled via the typical number of attached motors. Moreover these models can be related via simple mappings to other problems of nonequilibrium statistical physics which include e.g. surface growth problems [1, 2].

Here we discuss the most prominent model of this kind the so-called asymmetric simple exclusion process (ASEP), what is considered on a periodic chain of  $N$  sites and with  $M$  particles. For particlewise (pt) disorder particle  $i$  may hop to empty neighboring sites with rates  $p_i$  to the right and  $q_i$  to the left, where  $p_i$  and  $q_i$  are independent and identically distributed random variables. For sitewise (st) disorder the random hopping rates are assigned to given sites of the lattice. In the *totally asymmetric model*  $q_i = 0$ , for every  $i$ , whereas in the *partially asymmetric model*,  $q_i > p_i$  for a finite fraction of  $i$ .

A characteristic feature of the ASEP and its variants is their sensitivity to spatial inhomogeneities or quenched disorder of any kind. This is expressed in boundary induced phase transitions [7] or phase separated states caused by a single defect [8] or disordered lattices [9]. For

pt disorder some exact results are available [10, 11, 12], in particular for the totally asymmetric case. Depending on the extremal properties of the hopping rate distribution there is a dynamical phase transition in the system, separating a homogeneous state from a nonhomogeneous one, in which there is a macroscopic particle free region in front of the domain of occupied sites. For st disorder the analytical results are scarce [13] and most of our knowledge is based on Monte Carlo simulations [9, 14] and mean-field calculations [9, 15]. In the totally asymmetric model as the strength of disorder is increasing in the stationary state the particle density is changing from a homogeneous phase into a segregated-density phase, in which macroscopic regions with densities  $0 < \rho_c < 1 - \rho_c < 1$  coexist and the macroscopic current,  $J$ , generally decreasing with decreasing  $\rho$ . In the partially asymmetric model  $\rho_c$ , and at the same time  $J$  approaches zero in the thermodynamic limit. More precisely the stationary velocity  $v$  vanishes in a large ring of  $N$  sites as:

$$v \sim N^{-z}, \quad (1)$$

and the accumulated distance traveled by the particles,  $x$ , in time,  $t$ , is given by  $x \sim t^{1/z}$ , where  $z$  is the dynamical exponent.

In this letter we consider the partially asymmetric disordered model and will show that a vanishing current state can be encountered for pt disorder, too. We will derive an analytical expression for  $z_{pt}$  (pt disorder) and conjecture a simple relation for  $z_{st}$  (st disorder). We will also consider a new stationary state in which the average drift is zero and the system has a diffusive motion. We show that this new state for both types of disorder has an infinite randomness fixed point (IRFP) scenario. So far IRFP has only been observed for quantum[16] and stochastic models [17, 18] on disordered lattices.

Here we consider first the ASEP with pt disorder and a configuration is characterized in terms of the number of empty sites,  $n_i$ , in front of the  $i$ th particle. The sta-

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tionary weight of a configuration  $n_1, n_2, \dots, n_M$  is given by [11, 19]:  $f_N(n_1, n_2, \dots, n_M) = \prod_{\mu=1}^M g_{\mu}^{n_{\mu}}$ , where

$$g_{\mu} = \left[ 1 - \prod_{k=1}^M \frac{q_k}{p_k} \right]^{-1} \left[ \sum_{i=0}^{M-1} \frac{1}{p_{\mu-i}} \prod_{j=\mu+1-i}^{\mu} \frac{q_j}{p_j} \right] \quad (2)$$

provided  $p_i > 0$  for all particles. The stationary velocity is given by:

$$v = \frac{Z_{N-1,M}}{Z_{N,M}}, \quad Z_{N,M} = \sum_{n_1, n_2, \dots, n_M} f_N(\{n_{\mu}\}). \quad (3)$$

where in the summation  $\sum_{\mu=1}^M n_{\mu} = N - M$ . We are interested in the properties of the state in the thermodynamic limit, where we define the control-parameter as:

$$\delta = \frac{[\ln p]_{\text{av}} - [\ln q]_{\text{av}}}{\text{var}[\ln p] + \text{var}[\ln q]}, \quad (4)$$

such that for  $\delta > 0$  ( $\delta < 0$ ) the particles move to the right (left). Here and in the following we use  $[\dots]_{\text{av}}$  to denote averaging over quenched disorder and  $\text{var}[x]$  stands for the variance of  $x$ . In the following we restrict ourselves to the domain  $\delta \geq 0$ .

The consequences of these formulae for random hopping rates have been analyzed thoroughly if  $v$  is non-vanishing in the thermodynamic limit [10, 11, 12], in particular for the totally asymmetric model. For the partially asymmetric model we consider the maximal term,  $g_{\max} = \max\{g_{\mu}\} = g_{\mu^*}$ , which in Eq.(2) is dominated by a product,  $\prod_{a < j < b} q_j/p_j$ , where  $a < j < b$  is the largest region in which  $q_j > p_j$ . The probability of existence of this rare region is,  $P(b-a) \sim \exp(-\alpha(b-a))$ , therefore among  $M$  particles its typical size is  $b-a \sim \ln M$ . So we obtain:  $g_{\max} \sim \exp(\sigma(b-a)) \sim M^{\gamma}$ ,  $\gamma > 0$ . As a consequence, in the thermodynamic limit  $Z_{N,M}$  is dominated by that term, in which  $n_{\mu^*}$  is macroscopic and therefore the stationary velocity in Eq. (3) is given by:  $v = g_{\max}^{-1}$ .

The distribution of the quantities,  $g_{\mu}$ , for a fixed  $\mu$  and in the thermodynamic limit  $g_{\mu} = \lambda$  is, up to a prefactor, in the form of a Kesten-variable [20]. For large  $\lambda$  this distribution takes the form,  $P(\lambda) \sim \lambda^{-(1+1/z)}$ , where  $z$  is the positive root of the equation

$$\left[ (p/q)^{1/z} \right]_{\text{av}} = 1. \quad (5)$$

We argue that  $z_{pt} = z$ . For a large, but finite  $M$ , and for a given realization of disorder two variables,  $g_{\mu}$  and  $g_{\mu'}$ , have negligible correlations, provided  $|\mu - \mu'| > \xi$ , where  $\xi$  is proportional to the (finite) correlation length in the system. Consequently, the distribution of the  $g_{\mu}$  variables in a given sample has the same power-law asymptotics with the exponent in Eq.(5). Since the stationary velocity is the inverse of the largest  $g_{\mu}$ , the distribution of  $v$  for different samples is obtained from the statistics of extremes [21]. Here we use the result that the distribution of the maximum of independent random variables,

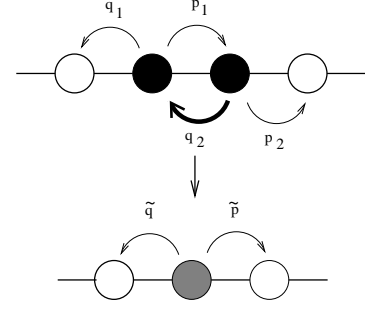


FIG. 1: Renormalization scheme for particle clusters. If  $q_2$  is the largest hopping rate, in a time-scale,  $\tau > 1/q_2$ , the two-particle cluster moves coherently and the composite particle is characterized by the effective hopping rates  $\tilde{q}$  and  $\tilde{p}$ , respectively, see the text.

which are taken from a distribution with a power-law tail is universal and given by the Fréchet distribution:  $P(u) = u^{-1-1/z}/ze^{-u^{-1/z}}$  with  $u = g_{\max}CM^{-z}$ . Thus  $v \sim M^{-z}$  and with  $M/N = O(1)$  we obtain from Eq. (1) the announced result. For small  $\delta$  the dynamical exponent is divergent:  $z \simeq (2\delta)^{-1}$ .

At  $\delta = 0$ , when the stationary velocity in the system is vanishing and the above formalism does not work, we use a strong disorder renormalization group (RG) approach, which is analogous to that applied recently for absorbing state phase transitions with quenched disorder [18] and originates in the theory of random quantum spin chains [16, 22] and random walks [17]. In the RG method one sorts the transition rates in descending order and the largest one sets the energy scale,  $\Omega = \max(\{p_i\}, \{q_i\})$ , which is related to the relevant time-scale,  $\tau = \Omega^{-1}$ . During renormalization the largest hopping rates are successively eliminated, thus the time-scale is increased. In a sufficiently large time-scale some cluster of particles moves coherently and form composite particles, which have new effective transition rates. To illustrate the method (see Fig. 1) let us assume that the largest rate is associated to a left jump, say  $\Omega = q_2$ , furthermore  $q_2 \gg p_2, q_1, p_1$ . In a time-scale,  $\tau > \Omega^{-1}$ , the fastest jump with rate  $q_2$  can not be observed and the two particles 1 and 2 form a composite particle. The composite particle has a left hopping rate  $\tilde{q} = q_1$ , since a jump of particle 1 is almost immediately followed by a jump of particle 2. The transition rate to the right,  $\tilde{p}$ , follows from the observation that, if the neighboring site to the right of particle 2 is empty it spends a small fraction of time:  $r = p_2/(p_2 + q_2) \approx p_2/q_2$  on it. A jump of particle 1 to the right is possible only this period, thus  $\tilde{p} = p_1 r \approx p_1 p_2 / q_2$ . The renormalization rules can be obtained similarly for a large  $p$ :

$$\tilde{p} = \frac{p_1 p_2}{\Omega}, \quad \Omega = q_2; \quad \tilde{q} = \frac{q_1 q_2}{\Omega}, \quad \Omega = p_1. \quad (6)$$

The RG scheme outlined above is completely equivalent to that of a random antiferromagnetic (dimerized)  $XX$

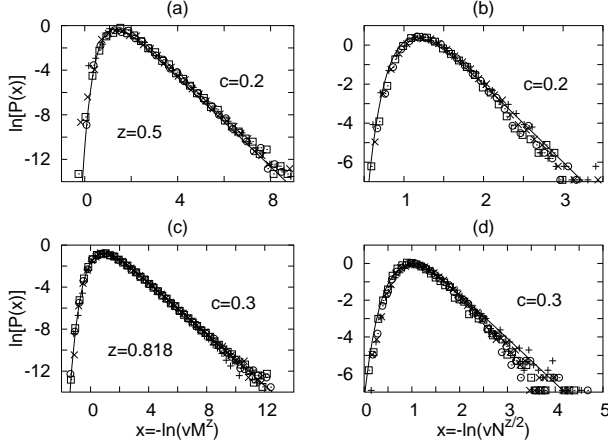


FIG. 2: Scaling of the velocity distribution in the Griffiths phase at two concentrations ( $c = 0.2$  and  $c = 0.3$ ) for pt (a,c) and st (b,d) disorder. Data denoted by symbols  $+$ ,  $\times$ ,  $\circ$ ,  $\square$  correspond to  $M = 64, 128, 256, 512$  for (a,c) and  $N = 64, 128, 256, 512$  for (b,d), respectively. The Fréchet distribution with the dynamical exponent:  $z = z_{pt}$ , calculated from Eq.(5) and with  $z_{st} = z_{pt}/2$ , as given in Eq.(8) are indicated by a full line.

spin chain of  $2M$  sites defined by the Hamiltonian:

$$H_{XX} = - \sum_{i=1}^{2M} J_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) , \quad (7)$$

with  $J_{2i-1} = p_i$  and  $J_{2i} = q_i$ . Here  $\delta$  in Eq. (4) plays the role of the dimerization. The strong disorder RG for the random XX-chain is analytically solved [23, 24] and the presumably asymptotically exact results[25] can be directly applied for the ASEP with pt disorder. In an extended part of the off-critical regime,  $\delta > 0$ , the correlation length is finite, but the typical time-scale,  $t_r$ , is divergent. This is the so-called Griffiths phase [27] in which several dynamical quantities, such as the susceptibility are singular. In the RG procedure in this phase almost exclusively the left hopping rates are decimated out [26]. After  $M$ -steps of decimation we are left with a single particle having a vanishing  $\tilde{q}/\tilde{p}$  and  $\tilde{p} \sim M^{-z}$ , with the dynamical exponent given in Eq. (5). This completely coincides with our previous results. At the critical point,  $\delta = 0$ , left and right hopping rates are decimated symmetrically and the system scales into an IRFP. After  $M$ -steps the remaining effective particle has a symmetric hopping probability:  $\tilde{q} \sim \tilde{p} \sim \exp(-const M^{1/2})$ . Thus the motion of the system is diffusive and ultra-slow, the appropriate scaling combination is given by:  $\ln v M^{-1/2}$ . Close to the critical point the correlation length in the system,  $\xi$ , which measures the width of the front, is given by  $\xi \sim \delta^{-2}$ .

These analytical results have been checked by calculating the velocity distribution of a periodic system with  $M/N = 1/2$ , using a bimodal distribution with  $p_i q_i = r$ , for all  $i$ , and  $P(p) = c\delta(p-1) + (1-c)\delta(p-r)$ , with  $r > 1$

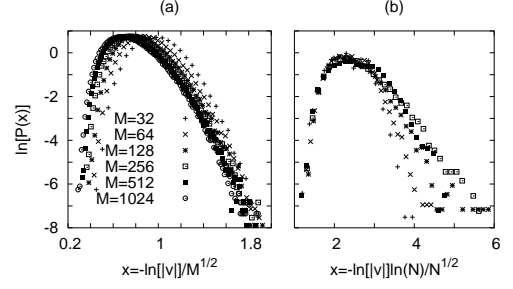


FIG. 3: Scaling plot of the velocity distribution at the critical point for pt (a) and st (b) disorder. In the latter case we have included a logarithmic correction term.

and  $0 < c \leq 1/2$ . In this case the control-parameter is  $\delta = (1 - 2c)/[2c(1 - c)\ln r]$  and the dynamical exponent from Eq. (5) is  $z = \ln r / \ln(c^{-1} - 1)$ . (In the limit  $c \ll 1/2$  one can make a direct calculation by noting that an extremely small velocity of  $v(l) = Cr^{-l}$  can be found in such a sample, in which  $l$  consecutive  $q$  rates have the value of  $r > 1$ . Such a sample (rare event) is realized with an exponentially small probability of  $P(l) \simeq c^l$ . Averaging over  $l$  we arrive to  $z \simeq -\ln r / \ln c$ .) As seen in Fig. 2ac in the Griffiths phase for two different values of the concentration,  $c < 1/2$ , the distributions are well described by the Fréchet statistics and the measured  $z$  agrees very well with the analytical result. At the critical point,  $c = 1/2$ , as shown in Fig. 3a, a scaling collapse is obtained in terms of the scaling variable,  $\ln v M^{-1/2}$ , which corresponds to the behavior at an IRFP.

For st disorder the rate of a jump to  $i \rightarrow i+1$  ( $i+1 \rightarrow i$ ) is  $p_i$  ( $q_i$ ) and we use the same distribution, as introduced before. In the small  $c$  limit  $z_{st}$  can be calculated along the lines presented for pt disorder, but now the velocity in a rare event of length  $l$  is [28]  $v(l) \simeq Ar^{-l/2+1/4}$ . The factor  $1/2$  in the exponential is due to the fact, that the boundary between the  $\rho = 1$  and  $\rho = 0$  phases lies in the middle of the unfavorable domain (particle-hole symmetry). For small concentration we obtain:

$$z_{st} = z_{pt}/2 . \quad (8)$$

According to our numerical investigations, as shown in Fig. 2bd, the relation in Eq. (8) seems to hold for not small values of  $c$ , too. In particular, at the critical situation,  $c = 1/2$ , we recover the result of the IRFP scenario, see Fig. 3b.

In the following we present arguments, why the relation in Eq. (8) can be generally valid. The stationary state both for pt and st disorder has a macroscopic phase separation. However, for pt disorder the occupied region slowly moves, which is due to single hole diffusion into the opposite direction, which takes place in a position dependent random potential, the hopping rates of which,  $p_i$  and  $q_i$ , are generated by the particles. The stationary velocity is the same as for a Sinai walker [29], see Eq. (5) and the relevant time-scale in the problem,  $\tau \sim v^{-1}$ , is given by the time needed for a single hole to

overcome the largest barrier (rare event) in the sample. This occurs with a probability of  $p_{pt}(\tau) \sim \tau^{-1/z_{pt}}$ , since the typical value of  $\tau = \tau_{pt}$  in a large sample is given by  $p_{pt}(\tau_{pt})N = 1$ , thus  $\tau_{pt} \sim N^{z_{pt}}$ , in accordance with Eq.(1).

On the other hand for the st disorder the position of the occupied block is fixed and the diffusion of holes through the occupied phase and diffusion of particles through the empty phase will result in the stationary current. In this case, due to particle-hole symmetry, the rare event consists of two (independent) large barriers, one for the holes and one for the particles, both having the same time-scale. The probability of occurrence of this rare event is  $p_{st}(\tau) \simeq p_{pt}^2(\tau)$ . Now the typical value of  $\tau = \tau_{st}$  is given by  $p_{st}(\tau_{st})N = 1$ , thus  $\tau_{st} \sim N^{z_{pt}/2} \sim N^{z_{st}}$ , from which Eq. (8) follows [30].

In summary the ASEP with sufficiently strong particle or lattice disorder has similar behavior. At  $\delta = 0$  in both cases the IRFP scenario holds, whereas for  $\delta > 0$ , in the Griffiths phase we have a singular dynamical behavior governed by a dynamical exponent, which is given in

Eqs. (5) and (8). The case of pt disorder can be described through RG transformation as an effective single-particle problem. For the st disorder case many particle effects seem to be important. Our findings might have possible applications, e.g. in the case of intracellular transport, which typically takes place on one-dimensional tracks. In case of uni-directional stochastic motion imperfections of the tracks generically lead to local effects, as far as physiological relevant particle concentrations are considered. By contrast we have shown that in case of bi-directional motion condensation of particles is generically observed in the presence of strong disorder, including the possibility of an effective control of the particle velocity according to Eq. (1).

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